JOURNAL OF APPROXIMATION THEORY **61**, 131–157 (1990)

On the Degree of Nonlinear Spline Approximation in Besov–Sobolev Spaces

P. OSWALD*

Sektion Mathematik, Technische Universität Dresden, 8027 Dresden, East Germany

Communicated by P. L. Butzer

Received February 18, 1986; revised November 26, 1987

This paper is devoted to estimating the degree of nonlinear spline approximation in Besov–Sobolev spaces defined on the unit cube in R^N. Good approximants with a given number of polynomial pieces and a given global smoothness are obtained from a certain decomposition of the functions under consideration into B-splines with respect to uniform dyadic partitions which, in turn, are constructed by means of a certain strategy of selecting terms with large coefficients. The general concept is applied to approximation by smooth splines with variable knots ($N=1$) and smooth nonlinear piecewise polynomial approximation with respect to partitions into cubes and certain triangulations ($N>1$).

INTRODUCTION

In this paper we describe a unified approach to estimating the degree of nonlinear spline approximation for Besov–Sobolev spaces in one and several dimensions.

First we give a short introduction to the problems under consideration. Let π be some partition of $[0, 1]^N$, $N \geqq 1$, into cells of given geometric structure, say, simplices or cubes, and denote by $|\pi|$ the number of cells in π. To any π we associate some linear set $S(\pi)$ of splines, e.g., piecewise polynomials with respect to π of given degree and global smoothness. For functions f: $[0, 1]^N \to R$ belonging to some Besov–Sobolev space $Y = B^s_{p,q}$ or W^s_p we consider asymptotic estimates of the nonlinear best approximations

$$e_n(f)_X = \inf_{\pi:|\pi| \leqq n} \inf_{g \in S(\pi)} \|f-g\|_X, \qquad n \to \infty,$$

* Present address: Sektion Mathematik, Friedrich-Schiller-Universität Jena, 6900 Jena, German Democratic Republic.

0021-9045/90 $3.00

in some other Besov–Sobolev norm $\|\cdot\|_X$. Roughly speaking, we shall study the degree of approximation by splines with "free partition" in Besov–Sobolev norms. Naturally, we assume $Y \subsetneq X$ and concentrate on those situations where choosing the partition into $\leqq n$ cells appropriately adapted to $f \in Y$ (as indicated in the definition of $e_n(f)_X$) substantially decreases the degree of approximation in comparison with the usual linear spline approximation methods where the partition is fixed for all $f \in Y$. For $N \geqq 1$ previous contributions to this area, especially to the case of nonsmooth piecewise polynomial approximation in the L_p norm $(1 \leqq p \leqq \infty)$, have been made by Birman and Solom'jak [3], Brudnyi [9, 10], Oskolkov [23], Rice and de Boor [7] *et al.*; for the one-dimensional case we also refer the reader to [11, 12, 27].

The basic idea which already appeared in [23] of our approach is to decompose functions $f \in B^s_{p,q}$ into a series of B-splines with respect to uniform dyadic partitions in such a way that the coefficients in this B-spline representation are controlled by the smoothness properties of the functions and vice versa. Then, good approximants satisfying $|\pi| \leqq n$ can easily be selected from this decomposition. To carry out this concept we only need the basic facts of linear approximation theory in L_p spaces, i.e., the direct and inverse theorems and properties of B-splines for uniform partitions. We also include the quasi-normed Besov spaces with $p<1$ which are important for some applications. The results on the degree of nonlinear spline approximation in L_p and Sobolev norms are obtained from those for Besov spaces by trivial embeddings.

A typical result that we can obtain in the one-dimensional case reads as follows. Let $m=1, 2, ..., 0<p<p' \leqq \infty$, $0<q$, $q' \leqq \infty$, $0<s<m$, and $0<s'<s-(1/p-1/p')$ be fixed. Then, for any function $f(x)$ belonging to the Besov space $B^s_{p,q}$ on $[0, 1]$ there exists a (smooth) spline $h^{(n)}$ of degree $m-1$ and defect 1 with less than n interior knots on $[0, 1]$ satisfying

$$\|f-h^{(n)}\|_{B^{s'}_{p',q'}} \leqq C \cdot n^{-(s-s')} \cdot \|f\|_{B^s_{p,q}} \tag{1}$$

and (if additionally $s<m+(1/p-1)$)

$$\|h^{(n)}\|_{B^s_{p,q}} \leqq C \cdot \|f\|_{B^s_{p,q}}, \tag{2}$$

where the constants are independent of f and $n=1, 2, ...$ (the corresponding definitions and more precise statements can be found in the following sections). For comparison, if we fix the partitions (for instance, if we consider the sequence of uniform partitions) no better estimate than $O(n^{-(s-s')+(1/p-1/p')})$ for $n \to \infty$ instead of (1) could be obtained. Let us also mention that in the case $0<p' \leqq p \leqq \infty$ $(0<s'<s)$ the estimates (1)

and (2) remain valid; however, this is a simple consequence of the linear theory (e.g., consider the uniform partitions of [0, 1]).

The method of proof of the above statement implies also the following result: With the same ranges of parameters m, p, p', and $1/p - 1/p' < s \leqq m$, for any given $f(x) \in \mathrm{Lip}(s, p) = B^s_{p,\infty}$ a sequence $\{h^{(n)}(x)\}$ of defect-1-splines of degree $m-1$ exists with less than n interior knots on [0, 1] satisfying the estimates

$$\|f - h^{(n)}\|_{L_{p'}} \leqq C \cdot n^{-s} \cdot \|f\|_{\mathrm{Lip}(s,p)}, \qquad n = 1, 2, ..., \tag{3}$$

and

$$\omega_m(t, h^{(n)})_p \leqq C \cdot \|f\|_{\mathrm{Lip}(s,p)} \cdot \begin{cases} t^s, & s < m - 1 + 1/p \\ t^s \cdot \ln 1/t, & s = m - 1 + 1/p \\ t^{m-1+1/p}, & s > m - 1 + 1/p \end{cases} \tag{4}$$

for $t \to 0$ with constants independent of n, t, and f (the definition of the moduli of continuity will be given in Section 1). While in (3) for the class $\mathrm{Lip}(s, p)$ the optimal degree of nonlinear spline approximation in the $L_{p'}$ metric is achieved (a fact which was already known) the inequality in (4) shows what order of smoothness in L_p will be preserved for the approximants.

In order to motivate the extension to Besov spaces with $p < 1$, let us consider the following example. Checking for the function

$$f(x) = x^a, \qquad x \in [0, 1],\ 0 < a < 1,$$

the modulus of continuity $\omega_2(t, f)_p$ it is easy to see that in the case $m = 2$, $1/2 < p < 1/(2-a)$, $p' = \infty$, we have $f \in \mathrm{Lip}(2, p)$ and thus the above statement yields the existence of linear splines $h^{(n)}(x)$ with less than n interior knots on [0, 1] such that $\|f - h^{(n)}\|_C = O(n^{-2})$, $n \to \infty$. Moreover, the corresponding optimal $O(n^{-(2-s')})$-estimate for the $\mathrm{Lip}(s', \infty)$ norm of $f - h^{(n)}$ also follows from our results ($0 < s' < a$). This remark shows, on the one hand, that function spaces related to the "pathological" L_p metric ($p < 1$) are connected with more classical problems in approximation theory (cf. also [9]). On the other hand, using this case previous results of several authors on improving the rate of approximation by splines of functions with typical singularities can be included here. However, our approach does not immediately lead to constructive algorithms such as the adaptive procedures proposed by J. Rice, de Boor *et al.*

We conclude this Introduction with a short outline of the following sections. In Section 1, we give the necessary definitions and state the properties of spline approximation schemes used in this paper. In particular, we prove the above-mentioned representation theorem for Besov spaces. In the next Section 2, the general statements on nonlinear approximation rates in Besov–Sobolev spaces ($0<p\leqq\infty$) are obtained. Finally, Section 3 is devoted to some situations where the results of Sections 1 and 2 apply, e.g., we consider the one-dimensional case, and the generation of nonlinear spline approximants in several dimensions by tensor product B-splines (partitions into cubes) and by box-splines on so-called type I partitions.

Applications of our approach to nonlinear approximation rates for functions with singularities and to error estimates for finite element methods will be discussed elsewhere (cf. [25]).

1

Let $I^N=[0,1]^N$ be the unit cube in R^N, and fix an integer $k=1,2,\ldots$. As usual, for $f(x)\in L_p(I^N)$ ($0<p\leqq\infty$) we denote by

$$\omega_k(t,f)_p=\sup_{|h|\leqq t}\left\{\int_{I^N_{h,k}}|\Delta_h^k f(x)|^p\,dx\right\}^{1/p},\qquad 0<t\leqq 1, \tag{5}$$

the (total) kth order L_p-modulus of continuity (cf. [30], here x, $h\in R^N$,

$$\Delta_h^k f(x)=\sum_{l=0}^{k}(-1)^{k-l}\cdot\binom{k}{l}\cdot f(x+lh),$$

and

$$I^N_{h,k}=\{x\in I^N\colon [x,x+lh]\subset I^N\}).$$

For this fixed notion of modulus of continuity (cf. Remark 5) we introduce the corresponding Besov spaces

$$B^s_{p,q}=\{f\in L_p(I^N)\colon \|f\|_{B^s_{p,q}}=\|f\|_{L_p}+|f|_{B^s_{p,q}}\},$$

where

$$|f|_{B^s_{p,q}}=\begin{cases}\left(\int_0^1(t^{-s}\cdot\omega_k(t,f)_p)^q\cdot t^{-1}\,dt\right)^{1/q}, & 0<q<\infty\\ \sup_{t\in(0,1]} t^{-s}\cdot\omega_k(t,f)_p, & q=\infty.\end{cases}$$

According to the well-known saturation properties of the moduli of continuity this definition makes sense if $0 \leqq s < k + \delta$ for $0 < q < \infty$ and $0 < s \leqq k + \delta$ for $q = \infty$ where $\delta = \delta(p) = \max(0, 1/p - 1)$ and $0 < p \leqq \infty$. In this way we obtain a scale of complete quasi-normed function spaces on I^N (Banach spaces if $1 \leqq p, q \leqq \infty$) which have been studied by many authors (cf. [2, 22, 31, 32]). For instance, by results of Triebel [31, 32] these spaces can, for $N\delta < s < k$, equivalently be introduced by

$$B^s_{p,q} = \{f \in \mathscr{D}'(I^N) \colon \exists g \in B^s_{p,q}(R^N) \text{ with } f = g|_{I^N}$$
$$\|f\|_{B^s_{p,q}} = \inf_{g \colon f = g|_{I^N}} \|g\|_{B^s_{p,q}(R^N)}\}$$

(with equivalent quasi-norms).

Thus, defining in analogy the usual Sobolev spaces

$$W^s_p = \{f \in \mathscr{D}'(I^N) \colon \exists g \in W^s_p(R^N) \text{ with } f = g|_{I^N}$$
$$\|f\|_{W^s_p} = \inf_{g \colon f = g|_{I^N}} \|g\|_{W^s_p(R^N)}\},$$

where $s \geqq 0$ and $0 < p < \infty$, and taking into account the well-known embedding theorems for Besov–Sobolev spaces we obtain

$$B^s_{p, \min(p, 2)} \subsetneq W^s_p \subsetneq B^s_{p, \max(p, 2)} \tag{6}$$

whenever $N\delta < s < k$, $0 < p < \infty$ (for further details, see [31, 32]). By this relation estimates in Sobolev norms can be derived from the corresponding ones for Besov spaces. Thus, we concentrate on the latter case.

Now we introduce a certain type of approximation scheme based on B-splines with respect to dyadic partitions and some of its basic properties. Let $S_0 \subset S_1 \subset S_2 \cdots$ be an increasing sequence of finite-dimensional subspaces of $L_\infty(I^N)$ and denote by $\{B_{l,i}, i = 1, ..., M_l\}$ a basis of S_l (playing the role of a B-spline basis). Furthermore, let us suppose that $M_l = \dim S_l \asymp 2^{Nl}$, $l = 0, 1, ...$. These notations and assumptions will be preserved throughout this paper.

The approximation scheme will satisfy the property $S(p)$, $0 < p \leqq \infty$, if we have

$$\|g\|_{L_p} \cdot C_1 \leqq 2^{-lN/p} \left(\sum_{i=1}^{M_l} |a_i|^p \right)^{1/p} \leqq C_2 \cdot \|g\|_{L_p}$$

with positive constants independent of l and $g = \sum_{i=1}^{M_l} a_i \cdot B_{l,i} \in S_l$ (the modification of the second term for $p = \infty$ is $\sup\{|a_i|, i = 1, ..., M_l\}$).

The approximation scheme $\{S_l\}$ possesses the approximation property $A(k, p)$ if for arbitrary $f \in L_p(I^N)$ the Jackson-type inequality

$$\inf_{g \in S_l} \|f-g\|_{L_p} \leqq C \cdot \omega_k(2^{-l}, f)_p, \qquad l=0, 1, \ldots,$$

holds with a constant independent of l and f.

The approximation scheme $\{S_l\}$ has the inverse property $I(k, p, \lambda)$, $0 \leqq \lambda \leqq k+\delta$, if for any $g \in S_l$ the Bernstein-type inequality

$$\omega_k(t, g)_p \leqq C \cdot (t \cdot 2^l)^\lambda \cdot \|g\|_{L_p}, \qquad 0<t \leqq 2^{-l},$$

holds with a constant independent of t, l, and g.

Remark 1. In the following we do not actually need the linear independence of the set $\{B_{l,i}\}$. Instead we might suppose that $S_l = \mathrm{lin}\{B_{l,i}, i=1, \ldots, M_l\}$, $l=0, 1, \ldots$, and modify the stability property $S(p)$ as follows: The upper estimate there is valid for some, while the lower estimate holds for any linear representation of $g \in S_l$ by the set $\{B_{l,i}\}$. Furthermore, as a rule, the approximation property $A(k, p)$ will be required not for all $f \in L_p(R^N)$ but only for functions belonging to the Besov space $B^s_{p,q}$ under consideration.

Examples of approximation schemes satisfying properties as defined above will be considered in Section 3.

We finish this section by proving a representation theorem with respect to an approximation scheme $\{S_l\}$ as introduced above. To simplify the notations we write in the following

$$\|a_{l,\cdot}\|_{l_p} = \begin{cases} \left(\sum_{i=1}^{M_l} |a_{l,i}|^p \right)^{1/p}, & 0<p<\infty \\ \max\{|a_{l,i}|, i=1, \ldots, M_l\}, & p=\infty \end{cases}$$

and introduce the sequence of quasi-norms $(s \geqq 0)$

$$\|b_l\|_{l^s_q} = \begin{cases} \left(\sum_{l=0}^{\infty} (2^{ls} \cdot |b_l|)^q \right)^{1/q}, & 0<q<\infty \\ \sup\{2^{ls} \cdot |b_l|, l=0, 1, \ldots\}, & q=\infty. \end{cases}$$

THEOREM 1. *Let $\{S_l\}$, $l=0, 1, \ldots$, be an approximation scheme as introduced above, and $0<q \leqslant \infty$.*

(a) *Suppose that, for some* $0<p\leqq\infty$, *the properties* $S(p)$ *and* $A(k,p)$ *hold. Then there exists for arbitrary* $f\in B^s_{p,q}$ *a representation*

$$f=\sum_{l=0}^{\infty} g_l=\sum_{l=0}^{\infty}\left(\sum_{i=1}^{M_l} a_{l,i}\cdot B_{l,i}\right) \qquad (\textit{convergence in } L_p) \tag{7}$$

such that (with a constant independent of f*)*

$$\|2^{-lN/p}\cdot\|a_{l,\cdot}\|_{l_p}\|_{l^s_q}\leqq C\cdot\|\,\|g_l\|_{L_p}\|_{l^s_q}\leqq C\cdot\|f\|_{B^s_{p,q}}. \tag{8}$$

(b) *Conversely, suppose that* $S(p)$ *and* $I(k,p,\lambda)$ *are satisfied where* $0<p\leqq\infty$, $0<s<\lambda$. *Then any (formal) representation* (7) *defines a function* $f\in B^s_{p,q}$ *where*

$$\|f\|_{B^s_{p,q}}\leqq C\cdot\|\,\|g_l\|_{L_p}\|_{l^s_q}\leqq C\cdot\|2^{-lN/p}\cdot\|a_{l,\cdot}\|_{l_p}\|_{l^s_q} \tag{9}$$

whenever the right-hand side of (9) *is finite.*

Proof. First consider part (a). For given $f\in B^s_{p,q}\subset L_p(I^N)$ we define functions $\bar g_l\in S_l$ such that

$$\|f-\bar g_l\|_{L_p}\leqq C\cdot\omega_k(2^{-l},f)_p,\qquad l=0,1,\ldots. \tag{10}$$

This follows by the property $A(k,p)$. Setting $g_l=\bar g_l-\bar g_{l-1}$, $l=1,2,\ldots$, and $g_0=\bar g_0$ we have $g_l\in S_l$. Thus, considering the basis representations

$$g_l=\sum_{i=1}^{M_l} a_{l,i}\cdot B_{l,i},\qquad l=0,1,\ldots,$$

we obtain by $S(p)$, (10), and the elementary properties of the modulus of continuity the estimates

$$\begin{aligned}2^{-lN/p}\cdot\|a_{l,\cdot}\|_{l_p}\leqq C\cdot\|g_l\|_{L_p}&\leqq C\cdot\begin{cases}\|f-\bar g_l\|_{L_p}+\|f-\bar g_{l-1}\|_{L_p}\\ \|f-\bar g_0\|_{L_p}+\|f\|_{L_p}\end{cases}\\ &=C\cdot\begin{cases}\omega_k(2^{-l},f)_p, & l=1,2,\ldots\\ \omega_k(2^{-l},f)_p+\|f\|_{L_p}, & l=0.\end{cases}\end{aligned}$$

Now inequality (8) follows immediately since

$$|f|_{B^s_{p,q}}\asymp\left\{\sum_{l=0}^{\infty}(2^{ls}\cdot\omega_k(2^{-l},f)_p)^q\right\}^{1/q}$$

(with the corresponding modification if $q=\infty$).

To prove part (b) we apply Minkowski's and Jensen's inequality ($\theta=\min(1,p)$) to obtain

$$(\|f\|_{B^s_{p,q}})^q \leqq C\cdot\left\{\left(\sum_{r=0}^{\infty}\|g_r\|^{\theta}_{L_p}\right)^{q/\theta}+\left(\sum_{l=0}^{\infty}2^{lsq}\left(\sum_{r=0}^{\infty}\omega_k(2^{-l},g_r)^{\theta}_p\right)^{q/\theta}\right)\right\}$$

$$\leqq C\cdot\sum_{l=0}^{\infty}2^{lsq}\left\{\sum_{r=0}^{l}2^{(r-l)\lambda\theta}\cdot\|g_r\|^{\theta}_{L_p}+\sum_{r=l+1}^{\infty}\|g_r\|^{\theta}_{L_p}\right\}^{q/\theta}$$

$$\leqq C\cdot\begin{cases}\displaystyle\sum_{l=0}^{\infty}2^{lsq}\left(\sum_{r=0}^{l}2^{(r-l)\lambda q}\|g_r\|^q_{L_p}+\sum_{r=l+1}^{\infty}\|g_r\|^q_{L_p}\right), & q\leqq\theta\\ \displaystyle\sum_{l=0}^{\infty}2^{lsq}\left\{\left(\sum_{r=0}^{l}2^{r\varepsilon}\right)^{q/\theta-1}\cdot\left(\sum_{r=0}^{l}2^{r(1-q/\theta)\varepsilon+(r-l)\lambda q}\cdot\|g_r\|^q_{L_p}\right)\right.\\ \displaystyle\qquad\left.+\left(\sum_{r=l+1}^{\infty}2^{-r\bar\varepsilon}\right)^{q/\theta-1}\left(\sum_{r=l+1}^{\infty}2^{-r\bar\varepsilon(1-q/\theta)}\cdot\|g_r\|^q_{L_p}\right)\right\},\end{cases}$$

where in the latter case ($\theta<q<\infty$) the real numbers $\varepsilon,\bar\varepsilon$ have been chosen according to the inequalities $0<\varepsilon<(\lambda-s)/(1/\theta-1/q)$ and $0<\bar\varepsilon<s/(1/\theta-1/q)$, resp. After changing the order of summation we finally get the first inequality in (9); the second one is obvious by the property $S(p)$. The remaining case $q=\infty$ can be handled in full analogy. Thus, the proof of Theorem 1 is complete.

Remark 2. It is easy to see that part (b) remains valid in the case $q\leqq\theta=\min(1,p)$ and also for $s=0$. Moreover, if we are interested in L_p-estimates of some (formal) representation (7) we require only property $S(p)$ in order to obtain

$$\|f\|_{L_p}\leqq C\cdot\|\ \|g_l\|_{L_p}\|_{l^0_\theta}\leqq C\cdot\|2^{-lN/p}\cdot\|a_{l,\cdot}\|_{l_p}\|_{l^0_\theta}. \tag{11}$$

Thus, in this case the inverse property will not be required.

2

In this section, for an approximation scheme $\{S_l\}$, $l=0,1,\ldots$, as introduced in Section 1, we consider the nonlinear best approximations

$$E_n(f,\{S_l\})_X=\inf\{\|f-g\|_X,\ g\in M_n(\{S_l\})\},\qquad n=1,2,\ldots, \tag{12}$$

where $M_n(\{S_l\})$ denotes the nonlinear set of finite sums $\sum_{l=0}^{\infty}\sum_{i=1}^{M_l}a_{l,i}\cdot B_{l,i}$ with $\leqq n$ nonzero coefficients $a_{l,i}$, and X stands for some Besov–Sobolev space. We shall see in Section 3 that in the applications to spline approximation schemes the study of the asymptotic behaviour of quantities (12) is closely related to that of $e_n(f)_X$.

First we introduce the construction of the nonlinear approximants from the set $M_n(\{S_l\})$. Our considerations correspond to the case $n \asymp 2^{Nr}$, $r = 0, 1, \ldots$ We fix $k = 1, 2, \ldots, 0 < p < \infty, 0 < s \leqq k + \delta$ ($\delta = \max(0, 1/p - 1)$), and denote $\alpha = -s + N/p$, $\beta = -N/p$. Let $\{S_l\}$, $l = 0, 1, \ldots$, be an approximation scheme, satisfying, at least, the properties $S(p)$ and $A(k, p)$. Thus, by Theorem 1(a) for $f \in B^s_{p,q}$, $0 < q \leqq \infty$, we can consider the representation (7) for which (8) holds true. Defining a new set of coefficients

$$\bar{a}^{(r)}_{l,i} = \begin{cases} a_{l,i} & \text{if } \; l < r \text{ or if } l \geqq r, \; |a_{l,i}| > c_0 \cdot 2^{\alpha l + \beta r} \\ 0 & \text{if } \; l \geqq r \text{ and } |a_{l,i}| \leqq c_0 \cdot 2^{\alpha l + \beta r} \end{cases} \tag{13}$$

we obtain by the (formal) representation

$$g^{(r)} = \sum_{l=0}^{\infty} g^{(r)}_l = \sum_{l=0}^{\infty} \sum_{i=1}^{M_l} \bar{a}^{(r)}_{l,i} \cdot B_{l,i}, \qquad r = 1, 2, \ldots, \tag{14}$$

a sequence of approximants. The constant $c_0 = c_0(f)$ appearing in (13) will be chosen below.

The following lemma shows that, actually, for $q = p$ the number $N^{(r)}$ of nonzero coefficients $\bar{a}^{(r)}_{l,i}$ in (13), (14) is finite.

LEMMA 1. *Under the above assumptions we have for* $f \in B^s_{p,p}$ $(s < k + \delta)$ *the estimate*

$$N^{(r)} \leqq C \cdot 2^{Nr} \cdot \left(1 + c_0^{-p} \cdot \sum_{l=r}^{\infty} 2^{lsp} \, \|g_l\|^p_{L_p}\right), \qquad r = 1, 2, \ldots. \tag{15}$$

Proof. Let, for any $l = 0, 1, \ldots$, $N^{(r)}_l$ be the number of nonzero coefficients $\bar{a}^{(r)}_{l,i}$, $i = 1, \ldots, M_l$. Then we have by (13) $N^{(r)}_l \leqq M_l \leqq C \cdot 2^{Nl}$, $l = 0, \ldots, r-1$, and

$$N^{(r)}_l \leqq c_0^{-p} \cdot 2^{-(\alpha l + \beta r)p} \cdot \|\bar{a}^{(r)}_{l,\cdot}\|^p_{l_p} \leqq C \cdot c_0^{-p} \cdot 2^{Nr + lsp} \cdot \|g_l\|^p_{L_p}$$

for $l = r, r+1, \ldots$. Since $N^{(r)} = \sum_{l=0}^{\infty} N^{(r)}_l$ this gives (15).

For the next lemmata we suppose that additionally $p < p' \leqq \infty$, $0 < q' \leqq \infty$, and $0 \leqq s' \leqq k + \delta'$ are given.

LEMMA 2. *Let* $p' < \infty$, $q = q' \cdot p/p'$. *Assume that in addition to* $S(p)$ *and* $A(k, p)$, *the approximation scheme* $\{S_l\}$ *satisfies* $S(p')$ *and* $I(k, p', \lambda')$. *Concerning* s' *we suppose that* $0 < s' < \lambda'$ *and* $s - s' \geqq N(1/p - 1/p')$. *Then for arbitrary* $f \in B^s_{p,q}$ *we have also* f, $g^{(r)} \in B^{s'}_{p',q'}$ *and*

$$\|f-g^{(r)}\|_{B^{s'}_{p',q'}} \leqq C\cdot 2^{-r(s-s')}\cdot c_0^{(1-p/p')}\cdot\begin{cases}\left(\sum_{l=r}^{\infty} 2^{lsq}\|g_l\|^q_{L_p}\right)^{1/q'} & 0<q'<\infty\\ (\sup_{l\geqq r} 2^{ls}\|g_l\|_{L_p}) & q'=\infty.\end{cases} \tag{16}$$

Proof. Here we have to use Theorem 1(b). Part (a) of Theorem 1 guarantees that the expressions on the right-hand side of (16) are finite. By $S(p)$, $S(p')$, and (13), (14) we obtain

$$\begin{aligned}\|g_l-g_l^{(r)}\|_{L_{p'}} &\leqq C\cdot 2^{-N/p'}\cdot\left(\sum_{i=1}^{M_l}|a_{l,i}-\bar{a}^{(r)}_{l,i}|^{p'}\right)^{1/p'}\\ &\leqq C\cdot(c_0\cdot 2^{\alpha l+\beta r})^{1-p/p'}\cdot 2^{-lN/p'}\cdot\left(\sum_{i=1}^{M_l}|a_{l,i}|^p\right)^{1/p'}\\ &\leqq C\cdot(c_0\cdot 2^{\alpha l+\beta r})^{1-p/p'}\cdot(\|g_l\|_{L_p})^{p/p'}, \qquad l=r, r+1, \ldots.\end{aligned}$$

Thus, according to (9) we get

$$\begin{aligned}\|f-g^{(r)}\|_{B^{s'}_{p',q'}} &\leqq C\cdot\|\,\|g_l-g_l^{(r)}\|_{L_{p'}}\|_{l^{s'}_{q'}}\\ &\leqq C\cdot(c_0\cdot 2^{\beta r})^{1-p/p'}\cdot(\|\{\|g_l\|_{L_p}, l\geqq r\}\|_{l^{\gamma}_{q}})^{p/p'},\\ &\gamma=s'p'/p+\alpha(p'/p-1)\end{aligned}$$

which immediately yields (16).

LEMMA 3. *Let $p'=\infty$ and $\{S_l\}$ satisfy the conditions of Lemma 2. Suppose that $0<q'\leqq\infty$ and $0<s'<\lambda'$ as well as $s-s'>N/p$ (with equality included if $q'=\infty$). Then for any $f\in B^s_{p,q}$, $0<q\leqq\infty$, we have $f, g^{(r)}\in B^{s'}_{\infty,q'}$ and*

$$\|f-g^{(r)}\|_{B^{s'}_{\infty,q'}}\leqq C\cdot c_0\cdot 2^{-r(s-s')}, \qquad r=1,2,\ldots. \tag{17}$$

Proof. By (13), (14), and $S(p')$ we have the estimate

$$\|g_l-g_l^{(r)}\|_{L_\infty}\leqq C\cdot c_0\cdot 2^{\alpha l+\beta r}, \qquad l=r, r+1, \ldots,$$

which, together with Theorem 1(b), directly yields (17).

Now we are in position to state the main results of this section.

THEOREM 2. *Let $0<p<p'\leqq\infty, 0<s<k+\delta$, $0<s'<\lambda'\leqq k+\delta'$, and suppose that the approximation scheme $\{S_l\}$ has the properties $S(p)$, $S(p')$, $A(k,p)$, and $I(k,p',\lambda')$. Then, for any function $f\in B^s_{p,p}$ and any $n\geqslant M_0$,* *there exist approximants* $h^{(n)} \in M_n(\{S_l\})$ *satisfying simultaneously in* $0 < s' \leqq s - N(1/p - 1/p')$ *the estimate*

$$\|f - h^{(n)}\|_{B^{s'}_{p',p'}} \leqq C \cdot n^{-(s-s')/N} \cdot \left(\int_0^{1/n} (t^{-s} \cdot \omega_k(t, f)_p)^p \cdot t^{-1}\, dt \right)^{1/p}, \quad (18)$$

where the constant C is independent of f and n.

Proof. It is clear that because of the asymptotic nature of (18) it will be sufficient to prove this inequality for some sequence $n = n_r \asymp 2^{Nr}$. For, we put in the construction of the approximants $g^{(r)}$ (cf. (13), (14)) the concrete value

$$c_0 = \left(\sum_{l=r}^{\infty} (2^{ls} \cdot \|g_l\|_{L_p})^p \right)^{1/p}$$

which only depends on $f \in B^s_{p,p}$ and r. Then, by Lemma 1 we get $g^{(r)} \in M_{N^{(r)}}(\{S_l\})$ with $N^{(r)} \asymp 2^{Nr}$, $r = 1, 2, \ldots$. Lemma 2 $(p' < \infty)$ and Lemma 3 $(p' = \infty)$ yield

$$\|f - g^{(r)}\|_{B^{s'}_{p',p'}} \leqq C \cdot c_0 \cdot 2^{-r(s-s')}, \qquad r = 1, 2, \ldots.$$

But by the definition of the g_l (cf. the proof of Theorem 1(a)) this actually implies (18) for $n = N^{(r)}$ and therefore also for general $n \geqslant M_0$. Thus, Theorem 2 is proved.

Remark 3. As a consequence of (18) we have

$$E_n(f, \{S_l\})_{B^{s'}_{p',p'}} = o(n^{-(s-s')/N}), \qquad n \to \infty, f \in B^s_{p,p}.$$

By the above constructions (or by interpolation arguments) we can obtain these estimates for $q \neq p$, $q' \neq p'$, too, sometimes restricting to the case $s' < s - N(1/p - 1/p')$. We state some further results in this direction; however, we concentrate on the formulations on O-bounds for the nonlinear best approximations (12).

THEOREM 3. *Let* $0 < p < p' \leqq \infty$, $0 < q, q' \leqq \infty$, *and* $\{S_l\}$ *satisfy the properties* $S(p)$, $S(p')$, $A(k, p)$, $I(k, p', \lambda')$. *Suppose that* $0 < s \leqq k + \delta$ *(with equality only for* $q = \infty$*), and* $0 < s' < \min(\lambda', s - N(1/p - 1/p'))$. *Then, for any* $f \in B^s_{p,q}$, *there exists a sequence* $h^{(n)} \in M_n(\{S_l\})$, $n \geqq M_0$, *such that*

$$\|f - h^{(n)}\|_{B^{s'}_{p',q'}} \leqq C \cdot n^{-(s-s')/N} \cdot \|f\|_{B^s_{p,q}}, \quad (19)$$

where C is independent of f and n.

Proof. We consider the case $p, p', q, q' < \infty$. The obvious modifications for the remaining cases are left to the reader. If $q \leqq \min(p, q'\, p/p')$ then the above construction can be used without substantial changes. Since $B^s_{p,q} \subsetneq B^s_{p,p}$ we get from Lemma 1

$$N^{(r)} \leqq C \cdot 2^{Nr} \cdot \left(1 + c_0^{-p} \cdot \left(\sum_{l=r}^{\infty} (2^{ls} \cdot \|g_l\|_{L_p})^q\right)^{p/q}\right), \qquad r=1,2,\dots,$$

and Lemma 2 yields

$$\|f-g^{(r)}\|_{B^{s'}_{p',q'}} \leqq C \cdot 2^{-r(s-s')} \cdot c_0^{1-p/p'} \cdot \left(\sum_{l=r}^{\infty} (2^{ls} \cdot \|g_l\|_{L_p})^q\right)^{p/(p'q)}.$$

Thus, choosing $c_0 = (\sum_{l=r}^{\infty} (2^{ls} \cdot \|g_l\|_{L_p})^q)^{1/q}$, we obtain the analogue of the o-estimate (18) in that case (moreover, $s' = s - N(1/p - 1/p')$ is allowed, and the sequence $h^{(n)}$ is independent of s', q, p', q').

In the general case we slightly modify the construction by setting $\alpha = -s + (N+\varepsilon)/p$, $\beta = -(N+\varepsilon)/p$ where $\varepsilon > 0$ will be a sufficiently small real number which will be chosen below. By the same arguments as in the proof of Lemma 1 we get

$$N^{(r)} \leqq C \cdot 2^{Nr} \cdot \left(1 + 2^{\varepsilon r} \cdot \left(\sum_{l=r}^{\infty} 2^{-l\varepsilon} \cdot (2^{ls} \cdot \|g_l\|_{L_p})^p\right) \cdot c_0^{-p}\right)$$

$$\leqq C \cdot 2^{Nr} \cdot \begin{cases} \left(1 + c_0^{-p} \cdot 2^{\varepsilon r} \cdot \left(\sum_{l=r}^{\infty} 2^{-l\varepsilon q/p} \cdot (2^{ls} \cdot \|g_l\|_{L_p})^q\right)^{p/q}\right), & q \leqq p \\ \left(1 + c_0^{-p} \cdot 2^{\varepsilon r} \cdot \left(\sum_{l=r}^{\infty} 2^{-l\varepsilon}\right)^{1-p/q} \cdot \left(\sum_{l=r}^{\infty} 2^{-l\varepsilon} \cdot (2^{ls} \cdot \|g_l\|_{L_p})^q\right)^{p/q}\right) & \\ & q > p \end{cases}$$

$$\leqq C \cdot 2^{Nr} \cdot \left(1 + c_0^{-p} \cdot \left(\sum_{l=r}^{\infty} (2^{ls} \cdot \|g_l\|_{L_p})^q\right)^{p/q}\right), \qquad r=1,2,\dots.$$

The analogous changes in the proof of Lemma 2 give the estimate

$$\|f-g^{(r)}\|_{B^{s'}_{p',q'}} \leqq C \cdot (c_0 \cdot 2^{\beta r})^{1-p/p'} \cdot \left(\sum_{l=r}^{\infty} 2^{-l\varepsilon'} \cdot (2^{ls} \cdot \|g_l\|_{L_p})^{q'p/p'}\right)^{1/q'},$$

where $\varepsilon' = q' \cdot (sp/p' - s' - \alpha \cdot (1-p/p')) = q' \cdot (s - s' - (N+\varepsilon)(1/p - 1/p'))$ is positive whenever ε is sufficiently small. Applying now the corresponding inequalities (for $q \leqq q'p/p'$ and $q > q'p/p'$, resp.) we obtain

$$\|f-g^{(r)}\|_{B^{s'}_{p',q'}} \leqq C \cdot 2^{-(s-s')r} \cdot c_0^{1-p/p'} \cdot \left(\sum_{l=r}^{\infty} (2^{ls} \cdot \|g_l\|_{L_p})^q\right)^{p/(p'q)}.$$

Thus, as above we get an estimate similar to (18) which obviously yields (19). Moreover, by choosing an appropriately small ε we see that the sequence $h^{(n)}$ can be defined independently of p', q', and $s' \in (0, s'_0]$ for any fixed $s'_0 < \min(\lambda', s - N(1/p - 1/p'))$. The proof of Theorem 3 is finished.

THEOREM 4. *Let* $1 \leqq p < p' < \infty$, $N(1/p - 1/p') < s \leqq k$, $0 \leqq s' < s - N(1/p - 1/p')$, *and let* $\{S_l\}$ *be an approximation scheme satisfying* $S(p)$, $S(p')$, $A(k, p)$, $I(k, p', \lambda')$ (*thus, additionally* $s' < \lambda'$). *Then, for* $f \in W^s_p$, *there exists a sequence* $h^{(n)} \in M_n(\{S_l\})$, $n \geqq M_0$, *such that*

$$\|f - h^{(n)}\|_{W^{s'}_{p'}} \leqq C \cdot n^{-(s-s')/N} \cdot \|f\|_{W^s_p}. \tag{20}$$

Proof. By the embedding theorem (6) we have $W^s_p \subsetneq B^s_{p,\max(p,2)}$ if $0 < s < k$ (obviously, $W^s_p \subsetneq B^s_{p,\infty}$ for $s = k$), and $B^{s'}_{p',\min(p',2)} \subsetneq W^{s'}_{p'}$ if $0 < s' < k$ (the case $s' = 0$, i.e., approximation rates in $L_{p'}$, is included in Theorem 5). Thus, applying Theorem 3, (19), with the corresponding q, q', we immediately obtain (20).

THEOREM 5. *Let* $0 < p < p' \leqq \infty$, $N(1/p - 1/p') < s \leqq k + \delta$, *and* $\{S_l\}$ *satisfy* $S(p)$, $S(p')$, $A(k, p)$. *Then, for any* $f \in B^s_{p,\infty}$, *a sequence* $h^{(n)} \in M_n(\{S_l\})$ *exists such that*

$$\|f - h^{(n)}\|_{L_{p'}} \leqq C \cdot n^{-s/N} \cdot \|f\|_{B^s_{p,\infty}}, \qquad n \geqq M_0. \tag{21}$$

Proof. Using Remark 2 and the construction as given in the proof of Theorem 3 (i.e., put $\alpha = -s + (N + \varepsilon)/p$, $\beta = -(N + \varepsilon)/p$ in (13) with appropriately small $\varepsilon > 0$) we get

$$\begin{aligned} \|f - g^{(r)}\|_{L_{p'}} &\leqq C \cdot \| \, \|g_l - g_l^{(r)}\|_{L_p} \|_{l^0_\theta} \\ &\leqq C \cdot 2^{-rs} \cdot c_0^{1 - p/p'} \cdot (\sup_{l \geqq r} 2^{ls} \cdot \|g_l\|_{L_p})^{p/p'}, \qquad r = 1, 2, ... \end{aligned}$$

as well as

$$N^{(r)} \leqq C \cdot 2^{Nr} \cdot (1 + c_0^{-p} \cdot (\sup_{l \geqq r} 2^{ls} \cdot \|g_l\|_{L_p})^p), \qquad r = 1, 2, ...$$

Now we can proceed as above.

Remark 4. The proof of Theorem 5 shows that for $f \in B^s_{p,q}$ ($q < \infty$) and $p' < \infty$ we can obtain also o-estimates similar to those in (18) for the $L_{p'}$-case.

THEOREM 6. *If, in addition to the assumptions of Theorems* 2, 3, *or* 5, *the approximation scheme satisfies the condition* $I(k, p, \lambda)$ *where* $0 < s < \lambda$, *then* (*with the corresponding* q)

$$\|h^{(n)}\|_{B^s_{p,q}} \leqq C \cdot \|f\|_{B^s_{p,q}}, \qquad n \geqq M_0.$$

This is an obvious consequence of Theorem 1 (b) and the construction (13), (14) of the sequence $g^{(r)}$ resp. $h^{(n)}$. Theorem 6 shows under what conditions the known smoothness of f in L_p can be preserved for the approximants. In the case of Theorem 5 we give a more precise statement concerning the order of the L_p modulus of continuity of the $h^{(n)}$:

$$\omega_k(t, h^{(n)})_p \leqq C \cdot \|f\|_{B^s_{p,\infty}} \cdot \begin{cases} t^s, & 0 < s < \lambda \\ t^s \cdot (\ln(1/t))^{1/\theta}, & s = \lambda \\ t^\lambda, & s > \lambda, \end{cases} \tag{23}$$

where $\theta = \min(1, p)$, $0 < t < \frac{1}{2}$, and the constant being independent of f and n. Indeed, by our construction of the approximants and $I(k, p, \lambda)$, $S(p)$ it follows that

$$\omega_k(t, g^{(r)})_p^\theta \leqq \sum_{l=0}^{\infty} \omega_k(t, g_l^{(r)})_p^\theta \leqq C \cdot \sum_{l=0}^{\infty} (\min(2^l t, 1))^{\lambda\theta} \cdot \|g_l\|_{L_p}^\theta$$

$$\leqq C \cdot (\|f\|_{B^s_{p,\infty}})^\theta \cdot \left(\sum_{2^l < 1/t} (t^\lambda \cdot 2^{l(\lambda - s)})^\theta + \sum_{2^l \geqq 1/t} 2^{-ls\theta} \right)$$

which yields (23).

Remark 5. Up to now, our considerations did not depend on the specific domain I^N as well as on the concrete definition of the (total) moduli of continuity $\omega_k(t, f)_p$ used in the definition of the Besov spaces and the properties $A(k, p)$, $I(k, p, \lambda)$ of the approximation schemes. For instance, all the statements of Sections 1 and 2 could be proved in the periodic case (i.e., for functions 1-periodic in each variable with the corresponding periodic moduli of continuity $\tilde{\omega}_k(t, f)_p = \sup_{|h| \leqq t} \|\Delta_h^k f\|_p$, $0 < t \leqq 1$). Another modification of this type where instead of $\omega_k(t, f)_p$ we shall use the sum of the partial (or coordinate) kth order moduli of continuity will be applied in Section 3(b).

Remark 6. Our abstract approach in the above sections should be considered as a generalization of paper [23] by Oskolkov who proved, in the situation of periodic piecewise linear approximation with respect to triangulations of I^2 into $< n$ triangles, the special case $2/p < s < 1$, $2 < p < p' \leqq \infty$, $k = 1$ of Theorem 5, and (23). For a certain extension of [23] see Sens [29].

3

In this section we describe some examples of concrete approximation schemes $\{S_l\}$ for which the constructions of the previous sections are applicable. To this end, we have to verify the properties $A(k, p)$, $I(k, p, \lambda)$, $S(p)$, and we have to discuss in more detail the connection between the nonlinear best approximations $E_n(f, \{S_l\})_X$ considered in Section 2 and the problems stated in the Introduction, i.e., the estimates for the quantities $e_n(f)_X$ which describe the degree of approximation by splines (corresponding to $\{S_l\}$) with "free partition."

(a) *Approximation by smooth splines with variable knots* ($N=1$). Given some partition

$$\pi: x^{(0)} = 0 < x^{(1)} < \cdots < x^{(n-1)} < x^{(n)} = 1$$

of $I = [0, 1]$ we define by

$$S_\pi^{(k)} = \{g \in C^{(k-2)}(I): g|_{I^{(i)}} \in P_k, i = 1, ..., n\}$$

the linear set of all polynomial splines of degree $k-1$ and minimal defect with respect to π. Here we have $k = 1, 2, ..., I^{(i)} = [x^{(i-1)}, x^{(i)})$, and P_k denotes the set of algebraic polynomials of degree $<k$. By

$$S_n^{(k)} = \bigcup_{\pi \in \Pi_n} S_\pi^{(k)}, \qquad n = 1, 2, ...,$$

where Π_n is the set of partitions with $<n$ interior knots in I, we denote the nonlinear set of splines of degree $k-1$ and minimal defect with $<n$ variable knots, and by

$$E_n^{(k)}(f)_X = \inf_{g \in S_n^{(k)}} \|f - g\|_X, \qquad n = 1, 2, ...,$$

the corresponding errors of best approximation of $f \in Y \subsetneq X$.

Evidently, studying the asymptotic behaviour of $E_n^{(k)}(f)_X$ we obtain estimates for the well-known problem of approximation by splines with $<n$ free knots (including multiple knots). There are many papers devoted to the latter problem for several norms $\|\cdot\|_X$ (especially for $X = L_p$ or C) and function classes Y as well as for individual functions f (cf., for instance, [11, 12, 28]).

In the present paper we are interested in the situation when $Y \subsetneq X$ are both Besov–Sololev classes. In order to apply the concept described above we introduce the dyadic partitions

$$\pi_l: t_0^{(l)} = 0 < t_1^{(l)} < \cdots < t_{2^l-1}^{(l)} < t_{2^l}^{(l)} = 1 \qquad (n = 2^l),$$

where $t_i^{(l)} = i \cdot 2^{-l}$, $i = 0, \pm 1, \ldots$. The standard B-spline basis of $S_l^{(k)}$ is given by the functions

$$N_{l,i}^{(k)}(x) = [t_i^{(l)}, \ldots, t_{i+k}^{(l)}, (x-t)_+^{k-1}] \cdot (t_{i+k}^{(l)} - t_i^{(l)}),$$

$i = -k+1, \ldots, 2^l - 1$ (for details we refer the reader to [27, 4]).

PROPOSITION 1. *The sequence* $\{S_{\pi_l}^{(k)}\}$, $l = 0, 1, \ldots$, *with the standard B-spline basis* $\{N_{l,i}^{(k)}(x); i = -k+1, \ldots, 2^l - 1\}$ *forms an approximation scheme in the sense of Section* 1 *satisfying the properties* $S(p)$, $A(k, p)$, *and* $I(k, p, \lambda)$ *for* $0 < p \leqq \infty$ *and* $\lambda = k - 1 + 1/p$.

Since this statement is known (for $1 \leqq p \leqq \infty$ cf. [27, 13], and the case $0 < p < 1$ has been considered in [24]) we omit the proof.

By the properties of the B-splines it is obvious that $E_{n(k+1)}^{(k)}(f)_X \leqq E_n(f, \{S_{\pi_l}^{(k)}\})_X$. Thus, the corresponding corollaries to Proposition 1 and the results of Section 2 are really estimates for the best approximation by splines of degree $k-1$ and minimal defect with variable knots. For instance, the results formulated in the Introduction follow from Theorems 3 and 6 resp. from Theorem 5 and (23). Moreover, by Theorem 2 and Proposition 1 we obtain

COROLLARY 1. *Let* $0 < p < p' \leqq \infty$, $0 < s < k + \delta$. *Then, for any* $f \in B_{p,p}^s$ *and* $n = 1, 2, \ldots$, *we can construct splines* $h^{(n)}(x)$ *of degree* $k-1$ *and minimal defect having* $< n$ *interior knots in* I *such that the inequality*

$$\|f - h^{(h)}\|_{B_{p',p'}^{s'}} \leqq C \cdot n^{-(s-s')} \left\{ \int_0^{1/n} \omega_k(t, f)_p^p \cdot t^{-sp-1} \, dt \right\}^{1/p}$$

holds, simultaneously, for $0 < s' \leqq s - (1/p - 1/p')$, $s' < \lambda' = k - 1 + 1/p'$ (*the constant* C *being independent of* f *and* n). *Additionally, if* $s < \lambda = k - 1 + 1/p$ *then*

$$\sup_n \|h^{(n)}\|_{B_{p,p}^s} \leqq C \cdot \|f\|_{B_{p,p}^s}.$$

The corresponding result for $s' = 0$, i.e. approximation in $L_{p'}$, has been proved by Brudnyi [10, Theorems 4 and 1] in a different way (and can also be obtained by our approach).

(b) *Smooth spline approximation with respect to partitions into cubes* ($N > 1$). This application is based on approximation schemes generated by tensor product B-splines.

Let $S_l^{(k)}(I^N) \subset C^{(k-2)}(I^N)$ ($\subset L_\infty(I^N)$ if $k = 1$) denote the class of smooth polynomial splines of coordinate degree $k-1$ (and minimal defect) with respect to the product partition $\pi_l^N = \pi_l \times \cdots \times \pi_l$ of I^N. The standard basis of $S_l^{(k)}(I^N)$ is given by the set of tensor product B-splines

$$\bar{N}_{l,i}^{(k)}(x) = \prod_{j=1}^{N} N_{l,i_j}^{(k)}(x_j), \qquad i = (i_1, ..., i_N),\ -k < i_j < 2^l.$$

The following proposition quotes the properties of this approximation scheme.

PROPOSITION 2. *Let $0 < p \leqq \infty$. Then sequence $\{S_l^{(k)}(I^N)\}$, $l = 0, 1, ...,$ satisfies property $S(p)$ while $A(k, p)$ and $I(k, p, k-1+1/p)$ are fulfilled in the following modified sense: for any $f \in L_p$ we have*

$$\inf_{g \in S_l^{(k)}(I^N)} \|f - g\|_{L_p} \leqq C \cdot \sum_{j=1}^{N} \omega_k^{(j)}(2^{-1}, f)_p \tag{24}$$

and for arbitrary $g \in S_l^{(k)}(I^N)$ there holds the inequality

$$\sum_{j=1}^{N} \omega_k^{(j)}(t, g)_p \leqq C \cdot (2^l \cdot t)^{k-1+1/p} \cdot \|g\|_{L_p}, \qquad 0 < t < 2^{-l}, \tag{25}$$

with constants depending on N, p, k, only. Here, by

$$\omega_k^{(j)}(t, f)_p = \sup_{0 \leqslant h_j < t} \left(\int_{I_{h,k}^N} |\Delta_h^k f(x)|^p \, dx \right)^{1/p}, \qquad 0 < t < 1$$

(where $h = (0, ..., h_j, ..., 0)$) we denote the partial modulus of continuity of order k with respect to the jth *coordinate of the function $f \in L_p(I^N)$.*

Proof. Inequality (24) has been proved for $1 \leqq p \leqq \infty$ in [20] by quasi-interpolant techniques and in [21] for general p via piecewise polynomial approximation and a certain smoothing procedure. The property $S(p)$ and (25) are essentially one-dimensional results. For instance, if $g \in S_l^{(k)}(I^N)$ then for any fixed $(x_2, ..., x_N) \in I^{N-1}$ the function $g_1(x_1) = g(x)$ belongs to the spline space $S_{\pi_l}^{(k)}$ with respect to $x_1 \in I$ and, therefore,

$$\|\Delta_{h_1}^k g_1\|_{L_p(0, 1-kh_1)}^p \leqq C \cdot (2^l \cdot h_1)^{(k-1)p+1} \cdot \|g_1\|_{L_p(I)}^p, \qquad 0 < h_1 < 2^{-l}.$$

Integrating this inequality with respect to $(x_2, x_3, ..., x_N) \in I^{N-1}$ we get, after taking the supremum over $h_1 \in (0, t]$,

$$\omega_k^{(1)}(t, g)_p \leqq C \cdot (2^l \cdot t)^{k-1+1/p} \cdot \|g\|_{L_p}, \qquad 0 < t < 2^{-l}.$$

Analogously, $S(p)$ follows from the one-dimensional property in Proposition 1.

In order to apply the results of the preceding sections together with Proposition 2 we shall make use of Remark 5. Replacing the total kth order modulus of continuity $\omega_k(t,f)_p$ in the definition of the Besov classes (cf. Section 1) by $\sum_{j=1}^N \omega_k^{(j)}(t,f)_p$ we obtain a new scale $\bar{B}^s_{p,q}$ which, however, at least for $N\delta < s < k$ coincides with the scale $B^s_{p,q}$. Since, obviously, the whole theory of Sections 1 and 2 can be carried over to the $\bar{B}^s_{p,q}$ spaces (without any new restrictions to the parameters) we are now in a position to state the corresponding corollaries as estimates for the quantities $E_n(f, \{S_l^{(k)}(I^N)\})_X$ or in terms of the nonlinear approximants $h^{(n)}$ (for $f \in Y \subsetneq X$, X, Y being two Besov–Sobolev spaces).

But in the N-dimensional case the partitions generated directly by the approximants $h^{(n)}$ are clearly more complicated and, in general, could be highly irregular (cf. Fig. 1 for a possible geometry of such a partition ($N=k=n=2$)). This situation is in contrast to what we would like to have, i.e., at least a locally well-structured partition. However, following the idea in [7] we can appropriately modify the considerations of Section 2 to obtain an estimate of the number of dyadic cubes of a certain properly nested partition containing the original partition corresponding to $h^{(n)}$ (see also [15]).

Let us introduce first the following terminology. The cubes

$$I_{l,i} = [(i_1-1)\cdot 2^{-1}, i_1\cdot 2^{-l}) \cdots [(i_N-1)\cdot 2^{-l}, i_N\cdot 2^{-l}), \qquad i \in Z^N$$

will be called dyadic of order $l = 0, 1, \ldots$. Any set of pairwise disjoint dyadic cubes $I_{l,i} \subset I^N$ where $\bigcup I_{l,i} = [0,1)^N$ defines a dyadic partition of I^N. Two dyadic cubes $I_{l,i}$ and $I_{l',i'}$ are neighbors if their closures have a nonempty intersection. A dyadic partition π is called properly nested if the order of two neighbors $I_{l,i}$ and $I_{l',i'}$ in π differs, at most, by 1, i.e., $|l-l'| \leq 1$ (for

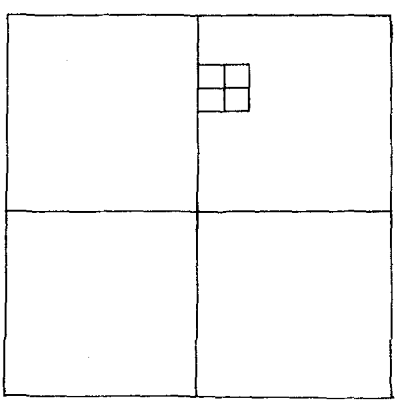

FIGURE 1

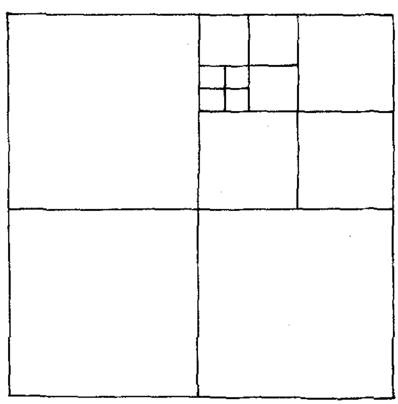

FIGURE 2

these definitions, cf. [7, 15]; Fig. 2 shows the smallest dyadic (but not properly nested) partition induced by that shown in Fig. 1). Finally, by $|\pi|$ we denote the number of cubes in π, and by $S_\pi^{(k)}(I^N)$ the class of smooth piecewise polynomials of coordinate degree $k-1$ with respect to the partition π (e.g., $S_l^{(k)}(I^N)$ is such a class generated by the uniform partition π_l^N consisting of all dyadic cubes of order l in I^N).

THEOREM 7. *Let* $0<p<p'\leqq\infty$, $0<q, q'\leqq\infty$, *and suppose that* $0<s\leqq k+\delta$ (*with equality only for* $q=\infty$), $0<s'<\min(s-N(1/p-1/p'), k+1/p'-1)$. *Then, for any* $f\in \bar{B}^s_{p,q}$, *we can construct a sequence* $\pi^{(n)}$ *of dyadic properly nested partitions with* $|\pi^{(n)}|\leqq n$ *and a sequence of approximants* $h^{(n)}\in S^{(k)}_{\pi^{(n)}}(I^N)$ *such that*

$$\|f-h^{(n)}\|_{\bar{B}^{s'}_{p',q'}}\leqq C\cdot n^{-(s-s')/N}\cdot\|f\|_{\bar{B}^s_{p,q}},\qquad n=1, 2, ..., \tag{26}$$

where C *is independent of* f *and* n. *Moreover, the partitions* $\pi^{(n)}$ *and the approximants* $h^{(n)}$ *can be chosen independently of* $s'\in(0, s_0']$ *for any fixed* $s_0'<\min(s-N(1/p-1/p'), k+1/p'-1)$.

Proof. For the approximation scheme $\{S_l^{(k)}(I^N)\}$ we shall use the construction in the proof of Theorem 3 and show the existence of a dyadic properly nested partition $\bar{\pi}^{(r)}$ consisting of $\leqq C\cdot 2^{Nr}\cdot(1+c_0^{-p}\cdot(\sum_{l=r}^\infty 2^{ls}\cdot\|g_l\|_{L_p})^q)^{p/q}$ cubes (cf. also Lemma 1) and such that $g^{(r)}\in S^{(k)}_{\bar{\pi}^{(r)}}(I^N)$ (here $r=1, 2, ...$ and $q<\infty$, with obvious changes if $q=\infty$). To this end, we first observe that the "sum" $\pi+\pi'$ of two dyadic properly nested partitions π and π' is once again properly nested and satisfies $|\pi+\pi'|\leqq|\pi|+|\pi'|$ (more precisely,

$$\pi+\pi'=\{I_{l,i}\text{: either } I_{l,i}\in\pi \text{ and } I_{l,i} \text{ does not contain smaller cubes from } \pi' \text{ or } I_{l,i}\in\pi' \text{ and it does not contain smaller cubes from } \pi\}).$$

We do not give a formal proof of this geometrically obvious fact (cf. Fig. 3 for an illustrative example if $N=2$).

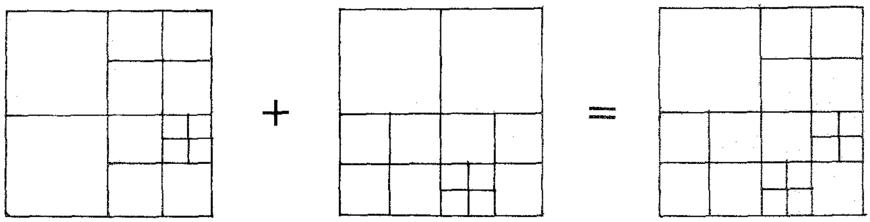

FIGURE 3

On the other hand, $S_{\pi}^{(k)}(I^N) \bigcup S_{\pi'}^{(k)}(I^N) \subset S_{\pi+\pi'}^{(k)}(I^N)$ and thus we can proceed as follows. Considering our approximant

$$g^{(r)}(x) = \sum_{l=0}^{\infty} g_l^{(r)}(x) = \sum_{l=0}^{\infty} \sum_i \bar{a}_{l,i} \cdot \bar{N}_{l,i}^{(k)}(x)$$

we construct to any pair (l, i) with $l \geqq r$, $\bar{a}_{l,i} \neq 0$, a minimal dyadic properly nested partition $\pi_{l,i}$ in such a way that $\bar{N}_{l,i}^{(k)}(x) \in S_{\pi_{l,i}}^{(k)}(I^N)$. Obviously, we can choose $\pi_{l,i}$ such that $\pi_{l,i} + \pi_{r-1}^N$ consists of $< A(N)$ cubes of order r, $r+1, ..., l$, resp., and $< 2^{N(r-1)}$ cubes of order $r-1$ where $A(N)$ is an appropriate absolute constant (cf. Fig. 4).

Thus, defining $\bar{\pi}^{(r)}$ by

$$\bar{\pi}^{(r)} = \sum_{l=r}^{\infty} \sum_{I:\, \bar{a}_{l,i} \neq 0} (\pi_{l,i} + \pi_{r-1}^N)$$

we obtain according to the above remarks $g^{(r)} \in S_{\bar{\pi}^{(r)}}^{(k)}(I^N)$ and

$$|\bar{\pi}^{(r)}| \leqq \sum_{l=r}^{\infty} A(N) \cdot (l-r+1) \cdot N_l^{(r)} + 2^{N(r-1)}.$$

Now, proceeding as in the proof of Theorem 3 (cf. Lemma 1, too) we get the required estimate for $|\bar{\pi}^{(r)}|$, and Theorem 7 is proved.

Remark 7. Analogous results to the other theorems of Section 2 can also be stated, e.g., o-estimates as given in Theorem 2 improving (26), estimates in L_p and Sobolev spaces (Theorems 4 and 5), or uniform estimates of the L_p-smoothness (in terms of the corresponding Besov classes) of the approximants $h^{(n)}$ for $s < k - 1 + 1/p$ as indicated in Theorem 6.

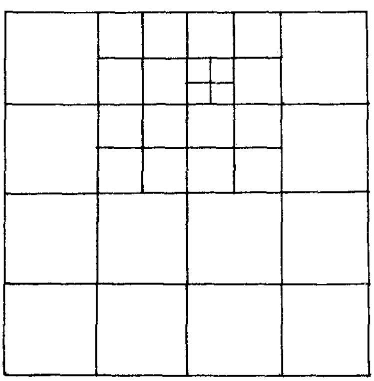

FIG. 4. The partition $\pi_{l,i}$ for $N = k = 2$, $l = 4$, and $(i_1, i_2) = (8, 2)$.

Concerning the above used facts of the modified Besov spaces $\bar{B}^s_{p,q}$ we refer the reader to the monographs [2, 22].

(c) *Nonlinear spline approximation generated by box splines on certain triangulations* ($N>1$). We concentrate on a simple case of box splines (corresponding to the so called Kuhn–Freudenthal or type I triangulation of R^N) which has been intensively studied for the last few years (cf. [5, 6, 8, 16–19, 14, 1]).

Let $e^1, ..., e^N$ be the unit vectors in R^N, and put $e^{N+1}=e^1+\cdots+e^N$. We consider any set $M=\{x^1,..., x^m\}$ of $m=m_1+\cdots+m_{N+1}$ vectors given by

$$M=\{\underbrace{e^1, ..., e^1}_{m_1 \text{ times}}, \underbrace{e^2, ..., e^2}_{m_2 \text{ times}},..., \underbrace{e^{N+1}, ..., e^{N+1}}_{m_{N+1} \text{ times}}\} \qquad (m_i \geqq 1).$$

By $B(x, M)$ we denote the standard box spline corresponding to M which can be introduced by the formulae

$$\int_{R^N} B(x, M)\cdot f(x)\,dx=\int_{I^m} f\left(\sum_{i=1}^{m} t_i\cdot x^i\right) dt, \qquad \forall f\in C_0(R^N).$$

The function $B(x, M)$ is a piecewise polynomial of total degree $m-N$ which belongs to $C^{d-1}(R^N)$ where $d=d(M)=\min\{(m_i+m_j)-1\colon i, j=1, ..., N+1,\ i\neq j\}$. Furthermore,

$$\operatorname{supp} B(\cdot, M)=\left\{x=\sum_{i=1}^{m} t_i\cdot x^i\colon t\in I^m\right\}.$$

The underlying partition corresponding to $B(x, M)$ (and its translates $B(x-v, M)$, $v\in Z^N$), the so called Kuhn–Freudenthal triangulation [19] of R^N, is produced by the hyperplanes

$$x=\sum_{i=1}^{N-1} s_i\cdot v^i+v, \qquad s_i\in R, \qquad v\in Z^N,$$

spanned by any set of $N-1$ different vectors v^i belonging to $\{e^1, ..., e^{N+1}\}$ (cf. Fig. 5 for the picture in the plane).

In the following we consider the approximation scheme

$$S_l^M(I^N)=\left\{g(x)=\sum_{i\in Z^N} a_i\cdot B(2^l\cdot x-i, M)|_{I^N},\ a_i\in R\right\}, \qquad l=0, 1, ...$$

It follows from known assertions on local linear independence of box splines (cf., e.g., [18]) that

$$\{B^M_{l,i}(x)=B(2^l\cdot x-i, M),\ i_j=-(m_j+m_{N+1})+1, ..., 2^l-1, j=1,..., N\}$$

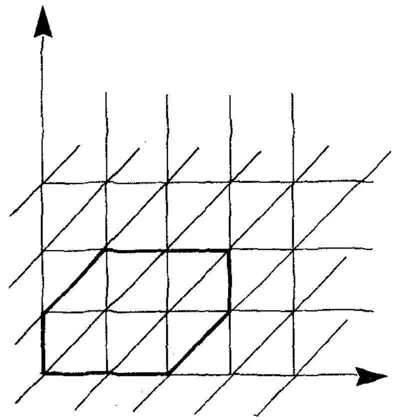

FIG. 5. The partition and supp $B(\cdot, M)$ in the case $N=2$, $m_1=2$, $m_2=m_3=1$, i.e., $m=4$, $d=1$, $m-N=2$ (thus, the corresponding box spline $B(x, M)$ is piecewise quadratic and belongs to $C(R^2)$).

forms a basis of $S_l^M(I^N)$, and that the approximation scheme has the property $S(p)$, $0<p\leqq\infty$. For $1\leqq p\leqq\infty$ this property as well as $A(d+1, p)$ also follows from the results in, e.g., [16]. Thus, we have to concentrate on $A(d+1, p)$ for $0<p<1$, and on the inverse property.

LEMMA 4. *For $0<p\leqq\infty$ and M as above we have*

$$\omega_{d+1}(t, B(\cdot, M))_{p, R^N} \leqq C \cdot t^{d+\min(1/p, 1)}, \qquad 0<t<1. \tag{27}$$

Proof. Using the formulas for the derivatives of box splines (cf. [5, 16]) and the definition of d we have

$$D_d B(x, M) \equiv \max_{|\alpha|=d} \left| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} B(x, M) \right| \leqq C < \infty \qquad \text{a.e. in } R^N$$

(more precisely, $B(x, M) \in W^d(RN)$), and, clearly,

$$D_{d+1} B(x, M) \begin{cases} =0, & x \in R^N \backslash \operatorname{supp} B(\cdot, M) \\ \leqq C, & x \in \operatorname{supp} B(\cdot, M) \cap \Omega_B, \end{cases}$$

where Ω_B is the open set given by the union of the interiors of the simplices forming the Kuhn–Freudenthal triangulation in R^N. These estimates yield, on the one hand,

$$\|\Delta_h^{d+1} B(\cdot, M)\|_{L_\infty(R^N)} \leqq 2 \cdot \|\Delta_h^d B(\cdot, M)\|_{L_\infty(R^N)} \leqq C \cdot |h|^d \tag{28}$$

and, on the other hand,

$$|\Delta_h^{d+1} B(x, M)| \leqq C \cdot |h|^{d+1}, \qquad x \in \Omega_{B,h}, \tag{29}$$

where $\Omega_{B,h}$ denotes the set of all those $x \in R^N$ for which the line segment $[x, x+(d+1)\cdot h]$ belongs to $\Omega_B \cap \operatorname{supp} B(\cdot, M)$. Since $\operatorname{mes}\{\operatorname{supp} \Delta_h^{d+1} B(\cdot, M)\} \leqq C$; $\operatorname{mes}\{(\operatorname{supp} \Delta_h^{d+1} B(\cdot, M)) \backslash \Omega_{B,h}\} \leqq C \cdot |h|$, we obtain by (28), (29)

$$\|\Delta_h^{d+1} B(\cdot, M)\|_{L_p(R^N)} \leqq C \cdot \left\{ \int_{\operatorname{supp} \Delta_h^{d+1} B(\cdot, M) \backslash \Omega_{B,h}} |h|^{dp} \, dx \pm \int_{\Omega_{B,h}} |h|^{(d+1)p} \, dx \right\}$$
$$\leqq C \cdot (|h|^{dp+1} + |h|^{(d+1)p})$$

for $|h| \leqq 1$ which implies the statement of Lemma 4.

Remark 8. In the one case $m_1 = \cdots = m_{N+1} = 1$ (i.e., the usual linear C^0-element, $d=1$) we have the better estimate

$$\omega_2(t, B(\cdot, M))_{p, R^N} \leqq C \cdot t^{1+1/p}, \qquad 0 < t < 1, \quad 0 < p \leqq \infty.$$

Obviously, Lemma 4 implies the inverse property $I(d+1, p, d+\min(1/p, 1))$. Concerning the approximation property $A(d+1, p)$ for $p<1$ we first use the Jackson type inequality (24) from [21] which yields (for $k=d+3$) the existence of piecewise polynomials $\bar{g}_l(x) \in S_l^{(d+3)}(I^N) \subset C^{d+1}(I^N)$ satisfying

$$\|f - \bar{g}_l\|_{L_p} \leqq C \cdot \sum_{j=1}^{N} \omega_{d+3}^{(j)} (2^{-l}, f)_p$$
$$\leqq C \cdot \omega_{d+3}(2^{-l}, f)_p \leqq C \cdot \omega_{d+1}(2^{-l}, f)_p$$

and, as a consequence,

$$\omega_{d+1}(2^{-l}, \bar{g}_1)_p \leqq C \cdot \omega_{d+1}(2^{-l}, f)_p, \qquad l = 0, 1, \ldots.$$

By usual one-dimensional arguments we also have

$$\omega_{d+1}(2^{-l}, \bar{g}_l)_p \geqq C \cdot 2^{-l(d+1)} \cdot \|D_{d+1} \bar{g}_l\|_{L_p}, \qquad l = 0, 1, \ldots$$

(all constants being independent of f and l).

Next we show the existence of functions $g_l \in S_l^M(I^N)$ such that

$$\|\bar{g}_l - g_l\|_{L_p} \leqq C \cdot 2^{-l(d+1)} \cdot \|D_{d+1} \bar{g}_l\|_{L_p}, \qquad l = 0, 1, \ldots. \tag{30}$$

To this end, we can use the quasi-interpolant theory for the C-metric.

According to [16], a sequence $g_l \in S_l^M(I^N)$ exists such that

$$\sup_{x \in I_{l,i}} |\bar{g}_l(x) - g_l(x)| \leqq C \cdot 2^{-l(d+1)} \cdot \sup_{x \in I'_{l,i}} D_{d+1} \bar{g}_l(x),$$

where $I'_{l,i} = \bigcup \{I_{l,j}\colon \operatorname{dist}(I_{l,j}, I_{l,i}) \leqq C(M) \cdot 2^{-l}\}$ with some absolute constant $C(M)$. For the definition of the dyadic cubes $I_{l,i}$, see the previous subsection (b).

From this local estimate we obtain

$$\begin{aligned} \|\bar{g}_l - g_l\|_{L_p}^p &\leqq \sum_{i\colon I_{l,i} \subset I^N} 2^{-Nl} \cdot (\sup_{x \in I_{l,i}} |\bar{g}_l(x) - g_l(x)|)^p \\ &\leqslant C \cdot 2^{-lp(d+1)} \cdot \sum_{i\colon I_{l,i} \subset I^N} 2^{-Nl} \cdot (\max_{I_{l,j} \subset I'_{l,i}} \sup_{x \in I_{l,j}} D_{d+1} \bar{g}_l(x))^p \\ &\leqq C \cdot 2^{-lp(d+1)} \cdot \sum_{i\colon I_{l,i} \subset I^N} 2^{-Nl} \cdot (\sup_{x \in I_{l,i}} D_{d+1} \bar{g}_l(x))^p \\ &\leqq C \cdot 2^{-lp(d+1)} \cdot \|D_{d+1} \bar{g}_l\|_{L_p}^p, \end{aligned}$$

where the last step follows from the trivial inequality

$$2^{-Nl} \cdot \|P\|_{L_\infty(I_{l,i})}^p \leqq C \cdot \|P\|_{L_p(I_{l,i})}^p, \qquad 0 < p < \infty,$$

satisfied for all polynomials $P(x)$ of fixed coordinate degree with a constant independent on l, i, and $P(x)$. Thus, (30) is established and $A(d+1, p)$ follows by

$$\begin{aligned} \|f - g_l\|_{L_p}^p &\leqq \|f - \bar{g}_l\|_{L_p}^p + \|\bar{g}_l - g_l\|_{L_p}^p \\ &\leqq C \cdot (\omega_{d+1}(2^{-l}, f)_p^p + 2^{-lp(d+1)} \cdot \|D_{d+1} \bar{g}_l\|_{L_p}^p) \\ &\leqq C \cdot \omega_{d+1}(2^{-l}, f)_p^p. \end{aligned}$$

We quote the above results in the following

PROPOSITION 3. *Let M be as described. Then, for $0 < p \leqq \infty$, the approximation scheme $\{S_l^M(I^N)\}$, $l = 0, 1, \ldots$, satisfies the properties $S(p)$, $A(d+1, p)$, and $I(d+1, p, d + \min(1, 1/p))$.*

Once again, Section 2 and Proposition 3 yield the corresponding assertions on the degree of approximation by linear combinations of $\leqslant n$ box splines $B_{l,i}^M(x)$. As in (b) we can suppose that the underlying triangulations of I^N are properly nested. The formulation of such corollaries is left to the reader.

(d) *Further comments.* All the statements in Sections 2 and 3 on the degree of nonlinear spline approximation for functions $f \in Y \subsetneq X$ where

$$Y = B^s_{p,q} \text{ or } W^s_p, \qquad X = B^{s'}_{p',q'} \text{ or } W^{s'}_{p'} \text{ (or } L_{p'},\ s'=0) \tag{31}$$

were given under the assumptions $0 < p < p' \leqq \infty$ and, in order to guarantee the embedding $Y \subsetneq X$, $s' < s - N(1/p - 1/p')$ (the other restrictions on the parameters s, s' reflect the properties of the approximation scheme while the whole range of q, q' is mainly used to include the theory for Sobolev classes as a particular case).

As for the other possible situation ($0 < p' \leqq p \leqq \infty$ and $s' \leqq s$) for which the embedding $Y \subsetneq X$ holds we can get analogous estimates as trivial consequences of the property $A(k, p)$. For example, to obtain the counterpart of Theorem 3 suppose that $0 < p' \leqq p \leqq \infty$, $0 < q, q' \leqq \infty$, $0 < s' < \min(s, \lambda')$, and $\{S_l\}$ satisfies $S(p)$, $S(p')$, $A(k, p)$, and $I(k, p', \lambda')$. For given $n \geqq M_0$ we define r by the condition $M_r \leqq n < M_{r+1}$ and define $h^{(n)} \in M_n(\{S_l\})$ as an element $\bar{g}_r \in S_r$ satisfying the Jackson-type inequality from $A(k, p)$ (cf. (10)). As in the proof of Theorem 1 we have

$$\begin{aligned} \|f - \bar{g}_r\|_{B^{s'}_{p',q'}} &\leqq C \cdot \|\{\|\bar{g}_l - \bar{g}_{l+1}\|_{L_p}, l \geqq r\}\|_{l^{s'}_{q'}} \\ &\leqq C \cdot \left(\sum_{l=r}^{\infty} (2^{ls'} \cdot \omega_k(2^{-l}, f)_p)^{q'} \right)^{1/q'} \\ &\leqq C \cdot 2^{-r(s-s')} \cdot \left(\sum_{l=r}^{\infty} (2^{ls} \cdot \omega_k(2^{-l}, f)_p)^{q} \right)^{1/q}. \end{aligned}$$

Since by definition $2^{rN} \asymp n$, we finally get

$$\|f - h^{(n)}\|_{B^{s'}_{p',q'}} \leqq C \cdot n^{-(s-s')/N} \cdot \left(\int_0^{1/n} (t^s \cdot \omega_k(t, f)_p)^q \cdot t^{-1}\, dt \right)^{1/q},$$

$n \geqq M_0$ (with obvious modifications if q or $q' = \infty$).

Thus, for $p' \leqq p$ the required estimate is really given (for any $f \in Y$) by the best approximation with respect to a fixed linear subspace of X with dimension $\leqq n$, and there is no reason to look for better approximants from the nonlinear set $M_n(\{S_l\})$.

The situation changes if $p < p'$. Clearly, the approximation error with respect to any fixed linear subspace $L \subset X$ with $\dim L \leqq n$ for functions belonging to Y can be estimated from below by the Kolmogorov n-width ($B(Y)$ denotes the unit ball in Y)

$$d_n(B(Y), X) = \inf_{L:\, \dim L \leqq n} \ \sup_{f \in B(Y)} \ \inf_{g \in L} \|f - g\|_X \tag{32}$$

which are well known in the case of normed Besov–Sobolev spaces (see [26, Theorem 2.2 (lower bounds)]). For example, let X, Y be as in (30), $1 \leqq q$, $q' \leqq \infty$, and $s' < s - N(1/p - 1/p')$. Then

$$d_n(B(Y), X) \geqq C \cdot \begin{cases} n^{-(s-s')/N}, & 2 \leqq p \leqq p' \leqq \infty \\ n^{-(s-s')/N + 1/2 - 1/p'}, & 1 \leqq p \leqq 2 \leqq p' \leqq \infty \\ n^{-(s-s')/N + 1/p - 1/p'}, & 1 \leqq p \leqq p' \leqq 2 \end{cases} \qquad (33)$$

for $n = 1, 2, \ldots$ (let us mention that the lower bound can be improved in some cases if we allow in definition (32) only subspaces L which are natural for the problems considered in the above sections).

Thus, when compared with (33) our theorems in Sections 2 and 3 show, in general, that for $p < p'$ nonlinear spline approximation with "free" partitions leads to stronger asymptotic estimates. Roughly speaking, under the corresponding assumptions on the parameters, $Y \subsetneq X$ as defined in (30), we have

$$\sup_{f \in B(Y)} E_n(f, \{S_l\})_X = \sup_{f \in B(Y)} \inf_{L = \operatorname{span}\{B_{l_1, i_1}, \ldots, B_{l_n, i_n}\}} \inf_{g \in L} \|f - g\|_X$$
$$\leqq C \cdot n^{-(s-s')/N}, \qquad n \to \infty. \qquad (34)$$

It should also be mentioned that nonlinear piecewise polynomial approximation has been used for estimating entropy numbers (see the classical paper [3] by Birman, Solom'jak).

REFERENCES

1. R. B. BARRAR AND M. L. LOEB, Some remarks on the exact controlled approximation order of bivariate splines on a diagonal mesh, *J. Approx. Theory* **42** (1984), 257–265.
2. O. V. BESOV, V. P. IL'IN, AND S. M. NIKOLSKII, "Integral Representations of Functions and Embedding Theorems," Nauka, Moskva, 1975. [Russian]
3. M. S. BIRMAN AND M. Z. SOLOM'JAK, Piecewise polynomial approximation of functions in W^s_p, *Mat. Sb.* (*N.S.*) **73** (1967), 331–355. [Russian]
4. C. DE BOOR, "A Practical Guide to Splines," Springer-Verlag, New York/Heidelberg/Berlin, 1978.
5. C. DE BOOR AND K. HÖLLIG, *B*-splines from parallelepipeds, *J. Analyse Math.* **42** (1982/83), 99–115.
6. C. DE BOOR AND K. HÖLLIG, Bivariate box splines and smooth pp. functions on a three-directional mesh, *J. Comput. Appl. Math.* **9** (1983), 13–28.
7. C. DE BOOR AND J. RICE, An adaptive algorithm for multivariate approximation giving optimal convergence rates, *J. Approx. Theory* **25** (1979), 337–359.
8. C. DE BOOR AND R. DEVORE, Approximation by smooth multivariate spines, *Trans. Amer. Math. Soc.* **276** (1983), 775–788.
9. JU. A. BRUDNYI, Piecewise polynomial approximation, embedding theorems and rational approximation, *in* "Approximation Theory" (R. Schaback and K. Scherer, Eds.), Lect. Notes Math., Vol. 556, pp. 73–98. Springer-Verlag, New York/Berlin, 1976.

10. JU. A. BRUDNYI, Rational approximation and embedding theorems, *Dokl. Akad. Nauk SSSR* **247** (1979), 269–272. [Russian]
11. H. G. BURCHARD AND D. F. HALE, Piecewise polynomial approximations on optimal meshes, *J. Approx. Theory* **14** (1975), 128–147.
12. H. G. BURCHARD AND D. F. HALE, On the degree of convergence of piecewise polynomial approximations on optimal meshes, *Trans. Amer. Math. Soc.* **234** (1977), 531–559.
13. Z. CIESIELSKI, Constructive function theory and spline systems, *Studia Math.* **53** (1975), 277–302.
14. C. K. CHUI AND R. H. WANG, Multivariate B-splines on triangulated rectangles, *J. Math. Anal. Appl.* **92** (1983), 533–551.
15. W. DAHMEN, Adaptive approximation by multivariate smooth splines, *J. Approx. Theory* **36** (1982), 118–140.
16. W. DAHMEN AND C. A. MICCHELLI, Translates of multivariate splines, *Linear Algebra Appl.* **52/53** (1983), 217–234.
17. W. DAHMEN AND C. A. MICCHELLI, Recent progress on multivariate splines, *in* "Approximation Theory IV," pp. 27–121, Academic Press, New York, 1983.
18. W. DAHMEN AND C. A. MICCHELLI, Some results on box splines, *Bull. Amer. Math. Soc. (N.S.)* **11** (1984), 147–150.
19. W. DAHMEN AND C. A. MICCHELLI, On the convexity of Bernstein polynomials and box spline surfaces, *Studia Sci. Math. Hungar.*, in press.
20. W. DAHMEN, R. DEVORE, AND K. SCHERER, Multidimensional spline approximation, *SIAM J. Numer. Anal.* **17** (1980), 380–402.
21. I. P. IRODOVA, Properties of functions defined by the rate of piecewise polynomial approximation, *in* "Studies in the Theory of Functions of Several Variables," pp. 92–117, Univ. of Jaroslavl, Jaroslavl, 1980. [Russian]
22. S. M. NIKOLSKII, Approximation of functions of several variables and embedding theorems, Springer-Verlag, New York/Heidelberg/Berlin, 1975.
23. K. I. OSKOLKOV, Polygonal approximation of functions of two variables, *Mat. Sb. (N.S.)* **107** (1978), 601–612. [Russian]
24. P. OSWALD, Spline approximation in the metric $L_p, 0<p<1$, *Math. Nachr.* **94** (1980), 69–96. [Russian]
25. P. OSWALD AND G. GEYMONAT, Some remarks on the approximation by finite element methods, *in* "Appr. Theory and Funct. Spaces," Banach Center Publ., in press.
26. A. PINKUS, "n-Widths in Approximation Theory," Springer-Verlag, New York/Berlin, 1985.
27. L. L. SCHUMAKER, "Spline Functions: Basic Theory," Wiley, New York, 1981.
28. L. L. SCHUMAKER, L_2-approximation by splines with free knots, *in* "Approx. in Theorie und Praxis" (G. Meinardus, Hrsg.), pp. 157–182, Bibl. Inst., Mannheim/Wien/Zürich, 1979.
29. F. M. SENS, Zur nichtlinearen Approximation mit polygonalen Funktionen bei freier Gitterwahl, *Wiss. Z. Tech. Univ. Dresden* **34** (1984), **4**, 83–86.
30. A. F. TIMAN, "Theory of Approximation of Functions of a Real Variable," Pergamon, Oxford/London/New York/Paris, 1963.
31. H. TRIEBEL, "Spaces of Besov–Hardy–Sobolev Type," Teubner-Texte Math., Vol. 15, Teubner, Leipzig, 1978.
32. H. TRIEBEL, "Theory of Function Spaces," Geest-Portig, Leipzig, 1983.